

# ON ROW SEQUENCES OF PADÉ AND HERMITE-PADÉ APPROXIMATION

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*Dedicated to my friend Ed on the occasion of his 70-th birthday*

**ABSTRACT.** A survey of direct and inverse type results for row sequences of Padé and Hermite-Padé approximation is given. A conjecture is posed on an inverse type result for type II Hermite-Padé approximation when it is known that the sequence of common denominators of the approximating vector rational functions has a limit. Some inverse type results are proved for the so called incomplete Padé approximants which may lead to the proof of the conjecture and the connection is discussed.

**Keywords** Montessus de Ballore Theorem · Simultaneous approximation · Hermite-Padé approximation · Rate of convergence · Inverse results

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## 1. INTRODUCTION

The study of direct and inverse type results for sequences of rational functions with a fixed number of free poles has been a subject of constant interest in the research of E.B. Saff. In different contexts (multi-point Padé [20], best rational [18]–[19], Hermite-Padé [12]–[14], and Padé orthogonal approximations [2]–[4]) such results are related with Montessus de Ballore's classical theorem [7] on the convergence of the  $m$ -th row of the Padé table associated with a formal Taylor expansion

$$(1) \quad f(z) = \sum_{n \geq 0} \phi_n z^n$$

provided that it represents a meromorphic function with exactly  $m$  poles (counting multiplicities) in an open disk centered at the origin, and its converse due to A.A. Gonchar [10, Section 3, Subsection 4], [11, Section 2] which allows to deduce analytic properties of  $f$  if it is known that the poles of the approximants converge with geometric rate.

Let  $m \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$  be fixed. If  $f$  is analytic at the origin,  $D_m(f)$  denotes the largest open disk centered at the origin to which  $f$  may be extended as a meromorphic function with at most  $m$  poles and  $R_m(f)$  is its radius; otherwise, we take  $D_m(f) = \emptyset$  and  $R_m(f) = 0$  for each  $m \in \mathbb{Z}_+$ .

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The value  $R_m(f)$  may be calculated, as shown by J. Hadamard [15], in terms of the Taylor coefficients  $\phi_n$ . Let  $\mathcal{P}_m(f)$  be the set of poles in  $D_m(f)$ . By  $(R_{n,m})_{n \geq 0, m \in \mathbb{Z}_+}$  fixed, we denote the  $m$ -th row of the Padé table associated with  $f$ , see Definition 1.2 restricted to  $d = 1$ .

The combined Montessus de Ballore–Gonchar theorem may be formulated in the following terms

**Theorem 1.1.** *Let  $f$  be a formal Taylor expansion about the origin and fix  $m \in \mathbb{N} = \{1, 2, \dots\}$ . Then, the following two assertions are equivalent.*

- a)  $R_0(f) > 0$  and  $f$  has exactly  $m$  poles in  $D_m(f)$  counting multiplicities.
- b) There is a monic polynomial  $Q_m$  of degree  $m$ ,  $Q_m(0) \neq 0$ , such that the sequence of denominators  $(Q_{n,m})_{n \geq 0}$  of the Padé approximations of  $f$ , taken with leading coefficient equal to 1, satisfies

$$\limsup_{n \rightarrow \infty} \|Q_m - Q_{n,m}\|^{1/n} = \theta < 1,$$

where  $\|\cdot\|$  denotes the  $\ell^1$  coefficient norm in the space of polynomials.

Moreover, if either a) or b) takes place, the zeros of  $Q_m$  are the poles of  $f$  in  $D_m(f)$ ,

$$(2) \quad \theta = \frac{\max\{|\xi| : \xi \in \mathcal{P}_m(f)\}}{R_m(f)},$$

and

$$(3) \quad \limsup_{n \rightarrow \infty} \|f - R_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(f)},$$

where  $K$  is any compact subset of  $D_m(f) \setminus \mathcal{P}_m(f)$ .

Since all norms in finite dimensional spaces are equivalent in b) any other norm in the  $m + 1$  dimensional space of polynomials of degree  $\leq m$  would do as well.

From Theorem 1.1 it follows that if  $\xi$  is a pole of  $f$  in  $D_m(f)$  of order  $\tau$ , then for each  $\varepsilon > 0$ , there exists  $n_0$  such that for  $n \geq n_0$ ,  $Q_{n,m}$  has exactly  $\tau$  zeros in  $\{z : |z - \xi| < \varepsilon\}$ . We say that each pole of  $f$  in  $D_m(f)$  attracts as many zeros of  $Q_{n,m}$  as its order when  $n$  tends to infinity.

Under assumptions a), in [7] Montessus de Ballore proved that

$$\lim_{n \rightarrow \infty} Q_{n,m} = Q_m, \quad \lim_{n \rightarrow \infty} R_{n,m} = f,$$

with uniform convergence on compact subsets of  $D_m(f) \setminus \mathcal{P}_m(f)$  in the second limit. In essence, Montessus proved that a) implies b), showed that  $\theta \leq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$ , and proved (3) with equality replaced by  $\leq$ . These are the so called direct statements of the theorem. The inverse statements, b) implies a),  $\theta \geq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$ , and the inequality  $\geq$  in (3) are immediate consequences of [10, Theorem 1]. The study of inverse problems when the behavior of individual sequences of poles of the approximants is known was suggested by A.A. Gonchar in [10, Subsection

12] where he presented some interesting conjectures. Some of them were solved in [21] and [22] by S.P. Suetin.

In [12], Graves-Morris and Saff proved an analogue of Montessus' theorem for Hermite-Padé (vector rational) approximation with the aid of the concept of polewise independence of a system of functions.

Let  $\mathbf{f} = (f_1, \dots, f_d)$  be a system of  $d$  formal or convergent Taylor expansions about the origin; that is, for each  $k = 1, \dots, d$ , we have

$$(4) \quad f_k(z) = \sum_{n=0}^{\infty} \phi_{n,k} z^n, \quad \phi_{n,k} \in \mathbb{C}.$$

Let  $\mathbf{D} = (D_1, \dots, D_d)$  be a system of domains such that, for each  $k = 1, \dots, d$ ,  $f_k$  is meromorphic in  $D_k$ . We say that the point  $\xi$  is a pole of  $\mathbf{f}$  in  $\mathbf{D}$  of order  $\tau$  if there exists an index  $k \in \{1, \dots, d\}$  such that  $\xi \in D_k$  and it is a pole of  $f_k$  of order  $\tau$ , and for  $j \neq k$  either  $\xi$  is a pole of  $f_j$  of order less than or equal to  $\tau$  or  $\xi \notin D_j$ . When  $\mathbf{D} = (D, \dots, D)$  we say that  $\xi$  is a pole of  $\mathbf{f}$  in  $D$ .

Let  $R_0(\mathbf{f})$  be radius of the largest open disk  $D_0(\mathbf{f})$  in which all the expansions  $f_k, k = 1, \dots, d$  correspond to analytic functions. If  $R_0(\mathbf{f}) = 0$ , we take  $D_m(\mathbf{f}) = \emptyset, m \in \mathbb{Z}_+$ ; otherwise,  $R_m(\mathbf{f})$  is the radius of the largest open disk  $D_m(\mathbf{f})$  centered at the origin to which all the analytic elements  $(f_k, D_0(f_k))$  can be extended so that  $\mathbf{f}$  has at most  $m$  poles counting multiplicities. The disk  $D_m(\mathbf{f})$  constitutes for systems of functions the analogue of the  $m$ -th disk of meromorphy defined by J. Hadamard in [15] for  $d = 1$ . Moreover, in that case both definitions coincide.

By  $\mathcal{Q}_m(\mathbf{f})$  we denote the monic polynomial whose zeros are the poles of  $\mathbf{f}$  in  $D_m(\mathbf{f})$  counting multiplicities. The set of distinct zeros of  $\mathcal{Q}_m(\mathbf{f})$  is denoted by  $\mathcal{P}_m(\mathbf{f})$ .

**Definition 1.2.** Let  $\mathbf{f} = (f_1, \dots, f_d)$  be a system of  $d$  formal Taylor expansions as in (4). Fix a multi-index  $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$  where  $\mathbf{0}$  denotes the zero vector in  $\mathbb{Z}_+^d$ . Set  $|\mathbf{m}| = m_1 + \dots + m_d$ . Then, for each  $n \geq \max\{m_1, \dots, m_d\}$ , there exist polynomials  $Q, P_k, k = 1, \dots, d$ , such that

- a.1)  $\deg P_k \leq n - m_k, k = 1, \dots, d, \quad \deg Q \leq |\mathbf{m}|, \quad Q \neq 0,$
- a.2)  $Q(z)f_k(z) - P_k(z) = A_k z^{n+1} + \dots$

The vector rational function  $\mathbf{R}_{n,\mathbf{m}} = (P_1/Q, \dots, P_d/Q)$  is called an  $(n, \mathbf{m})$  (type II) Hermite-Padé approximation of  $\mathbf{f}$ .

Type I and type II Hermite-Padé approximation were introduced by Ch. Hermite and used in the proof of the transcendence of  $e$ , see [16]. We will only consider here type II and, for brevity, will be called Hermite-Padé approximants.

In contrast with Padé approximation, such vector rational approximants, in general, are not uniquely determined and in the sequel we assume that given  $(n, \mathbf{m})$  one particular solution is taken. For that solution we write

$$(5) \quad \mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \dots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \dots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}},$$

where  $Q_{n,\mathbf{m}}$  is a monic polynomial that has no common zero simultaneously with all the  $P_{n,\mathbf{m},k}$ . Sequences  $(\mathbf{R}_{n,\mathbf{m}})_{n \geq |\mathbf{m}|}$ , for which  $\mathbf{m}$  remains fixed when  $n$  varies are called row sequences.

For each  $r > 0$ , set  $D_r = \{z \in \mathbb{C} : |z| < r\}$ ,  $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$ , and  $\overline{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$ .

**Definition 1.3.** Let  $\mathbf{f} = (f_1, \dots, f_d)$  be a system of meromorphic functions in the disk  $D_r$  and let  $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ . We say that the system  $\mathbf{f}$  is polewise independent with respect to  $\mathbf{m}$  in  $D_r$  if there do not exist polynomials  $p_1, \dots, p_d$ , at least one of which is non-null, such that

- b.1)  $\deg p_k < m_k$  if  $m_k \geq 1$ ,  $k = 1, \dots, d$ ,
- b.2)  $p_k \equiv 0$  if  $m_k = 0$ ,  $k = 1, \dots, d$ ,
- b.3)  $\sum_{k=1}^d p_k f_k$  is analytic on  $D_r$ .

In [12, Theorem 1], Graves-Morris and Saff established an analogue of the direct part of the previous theorem when  $\mathbf{f}$  is polewise independent with respect to  $\mathbf{m}$  in  $D_{|\mathbf{m}|}(\mathbf{f})$  obtaining upper bounds for the convergence rates corresponding to (2) and (3). It should be stressed that [12] was pioneering in the sense that it initiated a convergence theory for Hermite-Padé approximation.

The result [12, Theorem 1] does not allow a converse statement in the sense of Gonchar's theorem. Inspired in the concept of polewise independence, in [6] we proposed the following relaxed version of it.

**Definition 1.4.** Given  $\mathbf{f} = (f_1, \dots, f_d)$  and  $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$  we say that  $\xi \in \mathbb{C} \setminus \{0\}$  is a system pole of order  $\tau$  of  $\mathbf{f}$  with respect to  $\mathbf{m}$  if  $\tau$  is the largest positive integer such that for each  $s = 1, \dots, \tau$  there exists at least one polynomial combination of the form

$$(6) \quad \sum_{k=1}^d p_k f_k, \quad \deg p_k < m_k, \quad k = 1, \dots, d,$$

which is analytic on a neighborhood of  $\overline{D}_{|\xi|}$  except for a pole at  $z = \xi$  of exact order  $s$ . If some component  $m_k$  equals zero the corresponding polynomial  $p_k$  is taken identically equal to zero.

The advantage of this definition with respect to that of polewise independence is that it does not require to determine a priori a region where the property should be verified. Polewise independence of  $\mathbf{f}$  in  $D_{|\mathbf{m}|}(\mathbf{f})$  with respect to  $\mathbf{m}$  implies that  $\mathbf{f}$  has in  $D_{|\mathbf{m}|}$  exactly  $|\mathbf{m}|$  system poles (counting their order).

We wish to underline that if some component  $m_k$  equals zero, that component places no restriction on Definition 1.2 and does not report any benefit in finding system poles; therefore, without loss of generality one can restrict the attention to multi-indices  $\mathbf{m} \in \mathbb{N}^d$ .

A system  $\mathbf{f}$  cannot have more than  $|\mathbf{m}|$  system poles with respect to  $\mathbf{m}$  counting their order. A system pole need not be a pole of  $\mathbf{f}$  and a pole may not be a system pole, see examples in [6].

To each system pole  $\xi$  of  $\mathbf{f}$  with respect to  $\mathbf{m}$  one can associate several characteristic values. Let  $\tau$  be the order of  $\xi$  as a system pole of  $\mathbf{f}$ . For each  $s = 1, \dots, \tau$  denote by  $r_{\xi,s}(\mathbf{f}, \mathbf{m})$  the largest of all the numbers  $R_s(g)$  (the radius of the largest disk containing at most  $s$  poles of  $g$ ), where  $g$  is a polynomial combination of type (6) that is analytic on a neighborhood of  $\overline{D}_{|\xi|}$  except for a pole at  $z = \xi$  of order  $s$ . Set

$$R_{\xi,s}(\mathbf{f}, \mathbf{m}) := \min_{k=1, \dots, s} r_{\xi,k}(\mathbf{f}, \mathbf{m}),$$

$$R_{\xi}(\mathbf{f}, \mathbf{m}) := R_{\xi,\tau}(\mathbf{f}, \mathbf{m}) = \min_{s=1, \dots, \tau} r_{\xi,s}(\mathbf{f}, \mathbf{m}).$$

Obviously, if  $d = 1$  and  $(\mathbf{f}, \mathbf{m}) = (f, m)$ , system poles and poles in  $D_m(f)$  coincide. Also,  $R_{\xi}(\mathbf{f}, \mathbf{m}) = R_m(f)$  for each pole  $\xi$  of  $f$  in  $D_m(f)$ .

Let  $\mathcal{Q}(\mathbf{f}, \mathbf{m})$  denote the monic polynomial whose zeros are the system poles of  $\mathbf{f}$  with respect to  $\mathbf{m}$  taking account of their order. The set of distinct zeros of  $\mathcal{Q}(\mathbf{f}, \mathbf{m})$  is denoted by  $\mathcal{P}(\mathbf{f}, \mathbf{m})$ . We have (see [6, Theorem 1.4])

**Theorem 1.5.** *Let  $\mathbf{f}$  be a system of formal Taylor expansions as in (4) and fix a multi-index  $\mathbf{m} \in \mathbb{N}^d$ . Then, the following assertions are equivalent.*

- a)  $R_0(\mathbf{f}) > 0$  and  $\mathbf{f}$  has exactly  $|\mathbf{m}|$  system poles with respect to  $\mathbf{m}$  counting multiplicities.
- b) The denominators  $Q_{n,\mathbf{m}}$ ,  $n \geq |\mathbf{m}|$ , of simultaneous Padé approximations of  $\mathbf{f}$  are uniquely determined for all sufficiently large  $n$  and there exists a polynomial  $Q_{|\mathbf{m}|}$  of degree  $|\mathbf{m}|$ ,  $Q_{|\mathbf{m}|}(0) \neq 0$ , such that

$$\limsup_{n \rightarrow \infty} \|Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}\|^{1/n} = \theta < 1.$$

Moreover, if either a) or b) takes place then  $Q_{|\mathbf{m}|} \equiv \mathcal{Q}(\mathbf{f}, \mathbf{m})$  and

$$(7) \quad \theta = \max \left\{ \frac{|\xi|}{R_{\xi}(\mathbf{f}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{f}, \mathbf{m}) \right\}.$$

If  $d = 1$ ,  $R_{n,m}$  and  $Q_{n,m}$  are uniquely determined; therefore, Theorem 1.5 contains Theorem 1.1. The analogue of (3) is found in [6, Theorem 3.7]).

In the rest of the paper we wish to discuss the case when

$$(8) \quad \lim_{n \rightarrow \infty} Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}, \quad \deg Q_{|\mathbf{m}|} = |\mathbf{m}|, \quad Q_{|\mathbf{m}|}(0) \neq 0,$$

but the rate of convergence is not known in advance. Now the reference in the scalar case is a result by S.P. Suetin [22].

**Theorem 1.6.** *Assume that  $\lim_{n \rightarrow \infty} Q_{n,m}(z) = Q_m(z) = \prod_{k=1}^m (z - z_k)$  and*

$$0 < |z_1| \leq \dots \leq |z_N| < |z_{N+1}| = \dots = |z_m| = R.$$

*Then  $z_1, \dots, z_N$  are the poles of  $f$  in  $D_{m-1}(f)$  (taking account of their order),  $R_N(f) = \dots = R_{m-1}(f) = R$ , and  $z_{N+1}, \dots, z_m$  are singularities of  $f$  on the boundary of  $D_{m-1}(f)$ .*

When  $m = 1$  it is easy to see from the definition that  $Q_{n,1} = z - (\phi_n/\phi_{n+1})$  whenever  $\phi_{n+1} \neq 0$ . Therefore, Suetin's theorem contains the classical theorem of E. Fabry [8] which states that  $\lim_{n \rightarrow \infty} \phi_n/\phi_{n+1} = \zeta \neq 0$  implies that  $R_0(f) = |\zeta|$  and  $\zeta$  is a singular point of  $f$ .

Let us introduce the concept of system singularity of  $\mathbf{f}$  with respect to  $\mathbf{m}$ .

**Definition 1.7.** *Given  $\mathbf{f} = (f_1, \dots, f_d)$  and  $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$  we say that  $\xi \in \mathbb{C} \setminus \{0\}$  is a system singularity of  $\mathbf{f}$  with respect to  $\mathbf{m}$  if there exists at least one polynomial combination  $F$  of the form (6) analytic on  $D_{|\xi|}$  and  $\xi$  is a singular point of  $F$ .*

We believe that the following result holds.

**Conjecture.** *Assume that  $Q_{n,\mathbf{m}}$  is unique for all sufficiently large  $n$ , (8) takes place, and let  $Q_{|\mathbf{m}|}(\zeta) = 0$ . Then,  $\zeta$  is a system singularity of  $\mathbf{f}$  with respect to  $\mathbf{m}$ . If  $\zeta \in D_1(F)$ , for some polynomial combination  $F$  determines the system singularity of  $\mathbf{f}$  at  $\zeta$ , then  $\zeta$  is a system pole of  $\mathbf{f}$  with respect to  $\mathbf{m}$  of order equal to the multiplicity of  $\zeta$  as a zero of  $Q_{|\mathbf{m}|}$ .*

This conjecture applied to the scalar case reduces to Theorem 1.6.

In Section 2, we give a result similar to Theorem 1.6 for so called incomplete Padé approximation. Such approximants were introduced in [5] and used in [6] to prove Theorem 1.5. In the final section we describe some steps which may lead to the proof of the conjecture.

## 2. INCOMPLETE PADÉ APPROXIMANTS

Consider the following construction.

**Definition 2.1.** *Let  $f$  denote a formal Taylor expansion as in (1). Fix  $m \geq m^* \geq 1$ . Let  $n \geq m$ . We say that the rational function  $r_{n,m}$  is an incomplete Padé approximation of type  $(n, m, m^*)$  corresponding to  $f$  if  $r_{n,m}$  is the quotient of any two polynomials  $p$  and  $q$  that verify*

- c.1)  $\deg p \leq n - m^*, \quad \deg q \leq m, \quad q \neq 0,$
- c.2)  $q(z)f(z) - p(z) = Az^{n+1} + \dots$

Given  $(n, m, m^*), n \geq m \geq m^*$ , the Padé approximants  $R_{n,m^*}, \dots, R_{n,m}$  can all be regarded as incomplete Padé approximation of type  $(n, m, m^*)$  of  $f$ . From Definition 1.2 and (5) it follows that  $R_{n,\mathbf{m},k}, k = 1, \dots, d$ , is an incomplete Padé approximation of type  $(n, |\mathbf{m}|, m_k)$  with respect to  $f_k$ .

In the sequel, for each  $n \geq m \geq m^*$ , we choose one incomplete Padé approximant. After canceling out common factors between  $q$  and  $p$ , we write  $r_{n,m} = p_{n,m}/q_{n,m}$ , where, additionally,  $q_{n,m}$  is normalized as follows

$$(9) \quad q_{n,m}(z) = \prod_{|\zeta_{n,k}| \leq 1} (z - \zeta_{n,k}) \prod_{|\zeta_{n,k}| > 1} \left(1 - \frac{z}{\zeta_{n,k}}\right).$$

Suppose that  $q$  and  $p$  have a common zero at  $z = 0$  of order  $\lambda_n$ . Notice that  $0 \leq \lambda_n \leq m$ . From c.1)-c.2) it follows that

- c.3)  $\deg p_{n,m} \leq n - m^* - \lambda_n$ ,  $\deg q_{n,m} \leq m - \lambda_n$ ,  $q_{n,m} \neq 0$ ,  
 c.4)  $q_{n,m}(z)f(z) - p_{n,m}(z) = Az^{n+1-\lambda_n} + \dots$ .

where  $A$  is, in general, a different constant from the one in c.2).

From Definition 2.1 it readily follows that for each  $n \geq m$

$$(10) \quad r_{n+1,m}(z) - r_{n,m}(z) = \frac{A_{n,m} z^{n+1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}^*(z)}{q_{n,m}(z)q_{n+1,m}(z)},$$

where  $A_{n,m}$  is some constant and  $q_{n,m-m^*}^*$  is a polynomial of degree less than or equal to  $m - m^*$  normalized as in (9).

The first difficulty encountered in dealing with inverse-type results is to justify in terms of the data that the formal series corresponds to an analytic element around the origin which does not reduce to a polynomial. Set

$$R_m^*(f) = \left( \limsup_{n \rightarrow \infty} |A_{n,m}|^{1/n} \right)^{-1}, \quad D_m^*(f) = \{z : |z| < R_m^*(f)\}.$$

Let  $B$  be a subset of the complex plane  $\mathbb{C}$ . By  $\mathcal{U}(B)$  we denote the class of all coverings of  $B$  by at most a numerable set of disks. Set

$$\sigma(B) = \inf \left\{ \sum_{i=1}^{\infty} |U_i| : \{U_i\} \in \mathcal{U}(B) \right\},$$

where  $|U_i|$  stands for the radius of the disk  $U_i$ . The quantity  $\sigma(B)$  is called the 1-dimensional Hausdorff content of the set  $B$ . In the papers we refer to below, the only properties used of the 1-dimensional Hausdorff content follow easily from the definition. They are: subadditivity, monotonicity, and that the 1-dimensional Hausdorff content of a disk of radius  $R$  and a segment of length  $d$  are  $R$  and  $d/2$ , respectively.

Let  $(\varphi_n)_{n \in \mathbb{N}}$  be a sequence of functions defined on a domain  $D \subset \mathbb{C}$  and  $\varphi$  another function defined on  $D$ . We say that  $(\varphi_n)_{n \in \mathbb{N}}$  converges in  $\sigma$ -content to the function  $\varphi$  on compact subsets of  $D$  if for each compact subset  $K$  of  $D$  and for each  $\varepsilon > 0$ , we have

$$\lim_{n \rightarrow \infty} \sigma\{z \in K : |\varphi_n(z) - \varphi(z)| > \varepsilon\} = 0.$$

We denote this writing  $\sigma\text{-}\lim_{n \rightarrow \infty} \varphi_n = \varphi$  inside  $D$ .

Using telescopic sums, it is not difficult to prove the following (see [5, Theorem 3.4]).

**Lemma 2.2.** *Let  $f$  be a formal power series as in (1). Fix  $m$  and  $m^*$  nonnegative integers,  $m \geq m^*$ . Let  $(r_{n,m})_{n \geq m}$  be a sequence of incomplete Padé approximants of type  $(n, m, m^*)$  for  $f$ . If  $R_m^*(f) > 0$  then  $R_0(f) > 0$ . Moreover,*

$$D_{m^*}(f) \subset D_m^*(f) \subset D_m(f)$$

*and  $D_m^*(f)$  is the largest disk in compact subsets of which  $\sigma\text{-}\lim_{n \rightarrow \infty} r_{n,m} = f$ . Moreover, the sequence  $(r_{n,m})_{n \geq m}$  is pointwise divergent in  $\{z : |z| > R_m^*(f)\}$  except on a set of  $\sigma$ -content zero.*

We also have (see [6, Corollaries 2.4, 2.5])

**Lemma 2.3.** *Let  $f$  be a formal power series as in (1). Fix  $m \geq m^* \geq 1$ . Assume that there exists a polynomial  $q_m$  of degree greater than or equal to  $m - m^* + 1$ ,  $q_m(0) \neq 0$ , such that  $\lim_{n \rightarrow \infty} q_{n,m} = q_m$ . Then  $0 < R_0(f) < \infty$  and the zeros of  $q_m$  contain all the poles, counting multiplicities, that  $f$  has in  $D_m^*(f)$ .*

Suppose that  $\limsup_n |A_{n,m}|^{1/n} = 1$ . It is known, that there exists a regularizing sequence  $(A_{n,m}^*)_{n \geq m}$  such that:

- i)  $\lim_{n \rightarrow \infty} A_{n,m}^*/A_{n+1,m}^* = 1$ ,
- ii)  $\{\log(A_{n,m}^*/n!)\}$  is concave,
- iii)  $|A_{n,m}| \leq |A_{n,m}^*|$ ,  $n \in \mathbb{Z}_+$ ,
- iv)  $|A_{n,m}| \geq c|A_{n,m}^*|$ ,  $n \in \Lambda \subset \mathbb{Z}_+$ ,  $c > 0$  for an infinite sequence  $\Lambda$ .

The use of such regularizing sequences is well established in the study of singularities of Taylor series (see, for example, [1] and [17]). Its use was extended by S.P. Suetin in [22] to Padé approximation for proving Theorem 1.6. The proofs of [22, Lemmas 1, 2] (see also [23, Chapter 1]) may be easily adjusted to produce the following result for incomplete Padé approximation.

**Lemma 2.4.** *Let  $f$  be a formal power series as in (1). Fix  $m \geq m^* \geq 1$ . Assume that  $\lim_{n \rightarrow \infty} |A_{n,m}|^{1/n} = 1$ . For any  $\delta > 0$*

$$(11) \quad \max_{|z| \geq e^\delta} |p_{n,m}(z)/(A_{n,m}^* z^n)| = \mathcal{O}(1), \quad n \rightarrow \infty.$$

For every compact  $K \subset \{z : |z| < e^{-\delta}\} \setminus \mathcal{P}(f)$ ,

$$(12) \quad \max_K |(q_{n,m}f - p_{n,m})(z)/(A_{n,m}^* z^n)| = \mathcal{O}(1), \quad n \rightarrow \infty.$$

Assume that there exists a polynomial  $q_m$ ,  $\deg q_m = m$ ,  $q_m(0) \neq 0$ , such that

$$\lim_{n \rightarrow \infty} q_{n,m} = q_m.$$

Let  $f$  be holomorphic in some region  $G \supset D_m^*(f) \setminus \mathcal{P}(f)$ . Then, for every compact  $K \subset G$ , (12) takes place.

In the sequel  $\text{dist}(\zeta, B_n)$  denotes the distance from a point  $\zeta$  to a set  $B_n$ . Let  $\mathcal{P}_{n,m}(f) = \{\zeta_{n,1}, \dots, \zeta_{n,m_n}\}$  be the set of zeros of  $q_{n,m}$  and the points are enumerated so that

$$|\zeta_{n,1} - \zeta| \leq \dots \leq |\zeta_{n,m_n} - \zeta|.$$

We say that  $\lambda = \lambda(\zeta)$  points of  $\mathcal{P}_{n,m}$  tend to  $\zeta$  if

$$\lim_{n \rightarrow \infty} |\zeta_{n,\lambda} - \zeta| = 0, \quad \limsup_{n \rightarrow \infty} |\zeta_{n,\lambda+1} - \zeta| > 0.$$

By convention,  $\limsup_{n \rightarrow \infty} |\zeta_{n,\kappa} - \zeta| > 0$  for  $\kappa > \liminf_{n \rightarrow \infty} m_n$ .

**Theorem 2.5.** *Let  $f$  be a formal power series as in (1). Fix  $m \geq m^* \geq 1$ . Assume that  $0 < R_m^*(f) < +\infty$ . Suppose that*

$$\lim_{n \rightarrow \infty} \text{dist}(\zeta, \mathcal{P}_{n,m}(f)) = 0.$$

Let  $\mathcal{Z}_n(f)$  be the set of zeros of  $q_{n,m-m^*}^*$ . If  $|\zeta| > R_m^*(f)$ , then

$$(13) \quad \lim_{n \in \Lambda} \text{dist}(\zeta, \mathcal{Z}_n(f)) = 0$$

where  $\Lambda$  is any infinite sequence of indices verifying iv) in the regularization of  $(A_{n,m})_{n \geq m}$ . If  $|\zeta| < R_m^*(f)$ , then either (13) takes place or  $\zeta$  is a pole of  $f$  of order greater or equal to  $\lambda(\zeta)$ . If  $\lim_{n \rightarrow \infty} q_{n,m} = q_m$ ,  $\deg q_m = m$ ,  $q_m(0) \neq 0$ , and  $|\zeta| = R_m^*(f)$  then we have either (13) or  $\zeta$  is a singular point. If the zeros of  $q_m$  are distinct then at least  $m^*$  of them are singular points of  $f$  and lie in the closure of  $D_m^*(f)$ , those lying in  $D_m^*(f)$  are simple poles.

*Proof.* Without loss of generality, we can assume that  $R_m^*(f) = 1$ . The general case reduces to it with the change of variables  $z \rightarrow z/R_m^*(f)$ . Assume that  $|\zeta| \neq 1$  and  $\zeta$  is a regular point of  $f$  should  $|\zeta| < 1$ . Choose  $\delta > 0$  such that  $|\zeta| > e^\delta$  or  $|\zeta| < e^{-\delta}$  depending on whether  $|\zeta| > 1$  or  $|\zeta| < 1$ , respectively. Let  $q_{n,m}(\zeta_n) = 0$ ,  $\lim_{n \rightarrow \infty} \zeta_n = \zeta$ .

Evaluating at  $\zeta_n$ , using (11), if  $|\zeta| > 1$  or (12), when  $|\zeta| < 1$ , and taking iv) into account, it follows that

$$|p_{n,m}(\zeta_n)/(A_{n,m}\zeta_n^n)| \leq C_1, \quad n \geq n_0, \quad n \in \Lambda,$$

where  $C_1$  is some constant and  $\Lambda$  is the sequence of indices which appears in the regularization of  $(A_{n,m})_{n \geq m}$ . (In the sequel  $C_1, C_2, \dots$  denote constants which do not depend on  $n$ .) However, from (10) it follows that

$$p_{n,m}(\zeta_n)/(A_{n,m}\zeta_n^n) = -\zeta_n^{1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}^*(\zeta_n)/q_{n+1,m}(\zeta_n),$$

which combined with the previous inequality gives

$$|q_{n,m-m^*}^*(\zeta_n)| \leq C_2 |q_{n+1,m}(\zeta_n)|, \quad n \geq n_0, \quad n \in \Lambda.$$

Therefore, (13) takes place.

If  $|\zeta| = 1$  and  $\zeta$  is a regular point the proof of (13) is the same as for the case when  $|\zeta| < 1$ . In this case use (12) on a closed neighborhood of  $\zeta$  contained in  $G \supset D_m^*(f) \setminus \mathcal{P}(f)$ .

Now, assume that  $|\zeta| < 1$  and  $\limsup_{n \in \Lambda} \text{dist}(\zeta, \mathcal{Z}_n(f)) > 0$ . Then,  $\zeta$  is a singular point of  $f$ . Since  $D_m^*(f) \subset D_m(f)$  according to Lemma 2.2,  $\zeta$  must be a pole of  $f$ . Let  $\tau$  be the order of the pole of  $f$  at  $\zeta$ . Set  $w(z) = (z - \zeta)^\tau$  and  $F = wf$ . Notice that  $F(\zeta) \neq 0$ . Using (12) and iv), it follows that there exists a closed disk  $U_r$  centered at  $\zeta$  of radius  $r$  sufficiently small so that

$$(14) \quad \max_{U_r} |(q_{n,m}F - p_{n,m}w)(z)/(A_{n,m}z^n)| \leq C_3. \quad n \geq n_0, \quad n \in \Lambda.$$

Suppose that  $\tau < \lambda(\zeta)$ . Since  $\sigma - \lim_{n \rightarrow \infty} r_{n,m} = f$  (see Lemma 2.2), it follows that for each  $n \in \mathbb{Z}_+$  there exists a zero of  $\eta_n$  of  $p_{n,m}$  such that  $\lim_{n \rightarrow \infty} \eta_n = \zeta$ . Take  $r > 0$  sufficiently small so that  $\min_{U_r} |F(z)| > 0$ . Substituting  $\eta_n$  in (14), we have

$$|q_{n,m}(\eta_n)/(A_{n,m}\eta_n^n)| \leq C_4, \quad n \geq n_0, \quad n \in \Lambda,$$

and taking into account that (10) leads to

$$q_{n,m}(\eta_n)/(A_{n,m}\eta_n^n) = \eta_n^{1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}^*(\eta_n)/p_{n+1,m}(\eta_n),$$

we obtain

$$|q_{n,m-m^*}^*(\eta_n)| \leq C_5 |p_{n+1,m}(\eta_n)|, \quad n \geq n_0, \quad n \in \Lambda.$$

Since  $\limsup_{n \in \Lambda} \text{dist}(\zeta, \mathcal{Z}_n(f)) > 0$ , it follows that

$$(15) \quad \lim_{n \in \Lambda'} |p_{n+1,m}(\eta_n)| > 0.$$

for some subsequence  $\Lambda' \subset \Lambda$ .

The normalization (9) imposed on  $q_{n,m}$  implies that for any compact  $K \subset \mathbb{C}$  we have  $\sup_n \max_K |q_{n,m}(z)| \leq C_6$ . So, any sequence  $(q_{n,m})_{n \in I}$ ,  $I \subset \mathbb{Z}_+$ , contains a subsequence which converges uniformly on any compact subset of  $\mathbb{C}$ . This, combined with  $\sigma\text{-}\lim_{n \rightarrow \infty} r_{n,m} = f$  in  $D_m^*(f)$ , and the assumption that  $\tau < \lambda(\zeta)$  imply that there exists a sequence of indices  $\Lambda'' \subset \Lambda'$  such that  $\lim_{n \in \Lambda''} p_{n+1,m} = F_1$  uniformly on a closed neighborhood of  $\zeta$ , where  $F_1$  is analytic at  $\zeta$  and  $F_1(\zeta) = 0$  (see [9, Lemma 1] where it is shown that under adequate assumptions uniform convergence on compact subsets of a region can be derived from convergence in 1-dimensional Hausdorff content). This contradicts (15). Thus,  $\tau \geq \lambda(\zeta)$  as claimed.

To complete the proof recall that  $\deg q_{n,m-m^*}^* \leq m - m^*$  for all  $n \geq m$ . In particular, this implies that for each  $n \in \Lambda$  the set  $\mathcal{Z}_n(f)$  has at most  $m - m^*$  points. Each zero  $\zeta$  of  $q_m$  such that either  $|\zeta| > 1$  or  $|\zeta| \leq 1$  and is regular attracts a sequence of points in  $\mathcal{Z}_n(f)$ ,  $n \in \Lambda$ . This is clearly impossible if the total number  $M$  of such zeros of  $q_m$  exceeds  $m - m^*$ . So,  $M \leq m - m^*$ . The complement is made up of zeros of  $q_m$  which are singular and lie in the closure of  $D_m^*(f)$ . Those lying in  $D_m^*(f)$  are simple poles according to Lemma 2.3.  $\square$

### 3. SIMULTANEOUS APPROXIMATION

Throughout this section,  $\mathbf{f} = (f_1, \dots, f_d)$  denotes a system of formal power expansions as in (4) and  $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$  is a fixed multi-index. We are concerned with the simultaneous approximation of  $\mathbf{f}$  by sequences of vector rational functions defined according to Definition 1.2 taking account of (5). That is, for each  $n \in \mathbb{N}$ ,  $n \geq |\mathbf{m}|$ , let  $(R_{n,\mathbf{m},1}, \dots, R_{n,\mathbf{m},d})$  be a Hermite-Padé approximation of type  $(n, \mathbf{m})$  corresponding to  $\mathbf{f}$ .

As we mentioned earlier,  $R_{n,\mathbf{m},k}$  is an incomplete Padé approximant of type  $(n, |\mathbf{m}|, m_k)$  with respect to  $f_k$ ,  $k = 1, \dots, d$ . Thus, from Lemma 2.2

$$D_{m_k}(f_k) \subset D_{|\mathbf{m}|}^*(f_k) \subset D_{|\mathbf{m}|}(f_k), \quad k = 1, \dots, d.$$

**Definition 3.1.** A vector  $\mathbf{f} = (f_1, \dots, f_d)$  of formal power expansions is said to be *polynomially independent* with respect to  $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$  if there do not exist polynomials  $p_1, \dots, p_d$ , at least one of which is non-null, such that

- d.1)  $\deg p_k < m_k$ ,  $k = 1, \dots, d$ ,
- d.2)  $\sum_{k=1}^d p_k f_k$  is a polynomial.

In particular, polynomial independence implies that for each  $k = 1, \dots, d$ ,  $f_k$  is not a rational function with at most  $m_k - 1$  poles. Notice that polynomial independence may be verified solely in terms of the coefficients of the formal Taylor expansions defining the system  $\mathbf{f}$ . The system  $\mathbf{f}$  is polynomially independent with respect to  $\mathbf{m}$  if for all  $n \geq n_0$  the polynomial  $Q_{n,\mathbf{m}}$  is unique and  $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$ , see [6, Lemma 3.2].

An approach to the proof of the conjecture could be

- Remove the restriction in the last part of Theorem 2.5 that the zeros of  $q_m$  are distinct.
- Assuming (8), apply the improved version of Theorem 2.5 to the components of  $\mathbf{f}$ .
- Using polynomial combinations of the form (6) prove that each zero of  $Q_{|\mathbf{m}|}$  is a system singularity. It is sufficient to consider multi-indices of the form  $\mathbf{m} = (1, 1, \dots, 1)$  (see beginning of [6, Section 3] for the justification); then, (6) reduces to linear combinations.
- Prove the last part of the conjecture using the final statement of Lemma 2.3.

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